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## GALOISIAN ALGEBRA.

*Lehrbuch der Algebra.* Von Heinrich Weber. Erster Band. Pp. xvi + 654. (Braunschweig: Vieweg und Sohn, 1895.)

A "Treatise on Algebra" is rarely found to fulfil the promise of its title. It is too often a mere collection of problems and examples, thrown together without much regard to order or method; such theory as the book contains is often imperfect, and occasionally even incorrect; and no attempt is made to suggest the idea of an ordered system of algebra, which proceeds along natural lines of development.

Prof. Weber's treatise is a work of an entirely different stamp. It is designed upon a perfectly definite, well-considered plan; its foundations are laid with the utmost care and precision; and the reader is carried on from stage to stage until he is abreast of some of the most interesting, as well as the most recent, of mathematical discoveries. The work may be described, in general terms, as a treatise on ordinary algebra, with special reference to its arithmetical applications; with the addition, subsidiary to the main subject, but very important in itself, of the theory of groups. But in order to give anything like an adequate idea of the author's scope and method, it will be necessary to analyse the different parts of his book in some detail.

The introduction is entirely arithmetical; at the same time it is an indispensable prelude to all that is to follow. It contains the elements of the theory of multiplicities, a rigorous theory of rational and irrational numbers, and a proof of the continuity of real numerical magnitude. The demonstrations are mostly inspired by Dedekind; but it is shown that Cantor's procedure leads to equivalent results. It is to be specially observed that the definitions of rational fractions, ratios, irrational, negative, and complex numbers are entirely independent of any hypothesis about the existence of divisible concrete quantities; and similarly with regard to the statements and proofs of the various propositions. Perhaps in the whole range of mathematics no more abstract reasoning can be found than that by which the continuity of numerical magnitude has been established; and it is very instructive to compare the vague, illusive glimmering of the truth afforded by "intuition" with the precise and logical *Begriff* which has been developed by the persevering effort of mathematical speculation. This is one of the cases where

> "obstinate questionings
> Of sense and outward things"

have justified their stubbornness by leading to discoveries of the highest importance. Arithmetic, and consequently the whole of analysis, has now been absolutely and finally freed from all necessity of appealing to theories or assumptions foreign to its own nature; and the effect of this liberation is already showing itself in many different ways. Thus, for example, in the first book of Prof. Weber's work will be found (pp. 101–126) a strictly arithmetical proof of the fundamental proposition that every algebraical equation in one variable with numerical coefficients has at least one real or complex numerical root. It is true that the proof is accompanied by a geometrical figure, but this is merely for the sake of convenience, and does not affect the real nature of the demonstration. The same thing applies to the proof (pp. 132-6) that the roots of an equation are continuous functions of its coefficients.

The first Book is to a great extent preparatory. It treats, in order, of rational functions of one or more variables, determinants, the existence of roots of algebraic equations, symmetric functions, invariants and covariants, and Tschirnhausen's transformation. The portions most worthy of remark are the proof of the existence of roots as numerical quantities, already alluded to, and the chapter on the Tschirnhausen transformation, which gives a very clear account of Hermite's modified form of the process, by means of which the coefficients of the transformed equation are expressible as simultaneous invariants of the original quantic $f(z)$ and a certain auxiliary quantic $T(z)$. The method is applied to the reduction of the general quintic to the normal forms which are associated with the names of Bring (or Jerrard) and Brioschi.

Book II. deals with the roots of algebraical equations, and comprises six chapters. Of these the first discusses the reality of the roots. The solution of quadratic, cubic, and biquadratic equations is followed by the proof of an important property of the Bezoutiant, namely, that if, by a real linear transformation, the Bezoutiant of $f(z)$ is expressed as a sum of $\pi$ positive and $\nu$ negative squares which cannot be reduced to a smaller number, then $\pi + \nu + 1$ is the number of distinct roots of $f(z) = 0$, and of these $\pi - \nu + 1$ are real, and the rest imaginary. (It is understood, of course, that all the coefficients of $f(z)$ are real.) This leads to a digression on Sylvester's Law of Inertia of quadratic forms; after which an application of the theorem about the Bezoutiant is made to the general cubic and biquadratic, and to two special quintic equations.

The Law of Inertia itself is proved in Book I. (p. 183-4): it may perhaps be remarked that on p. 184, line 17, the phrase, "aus diesen $\mu$ Gleichungen" is not very clear. What is really meant is the system of $\mu$ linear equations, by means of which (*ex hypothesi*) $Z_1, Z_2 \ldots Z_\mu$ are expressible in terms of $Y_1, Y_2 \ldots Y_\nu, {Z^1}_1, {Z^1}_2 \ldots {Z^1}_{\mu 1}$. The supplementary articles on the law of inertia (pp. 255–265) explain how the characteristic numbers $\pi$, $\nu$ for a given quadratic form may be deduced from the coefficients of the form.

Chapter viii., which next follows, contains an account of Sturm's Theorem which is very fresh and interesting, and includes Hermite's determinants, which may be used instead of the Sturmian functions proper. After this comes a sketch of Kronecker's remarkable theory of characteristics, and an application of it to Gauss's first proof of the existence of roots of equations.

Chapters ix. and x. treat of the separation and approximate calculation of roots. On the whole they follow the traditional lines; and we confess that we did not find them so interesting as the rest of the work. From the author's own point of view, they are, in a sense, superfluous; and, in fact, no use is made of them subsequently. Then from the practical point of view it is hardly satisfactory to fill up three pages with an account of Daniel Bernoulli's method of approximation, and omit all mention of Horner's algorithm. Still these chapters are redeemed from commonplace by a very elementary proof of Newton's Rule (first demonstrated by Sylvester), a geometrical excursus, after Klein, and some very curious theorems of Laguerre's relating to equations with no imaginary roots.

Chapter xi. discusses continued fractions, arithmetical equivalence, and the theory of the reduction of quadratic irrational numbers. A quadratic irrational $\{a, b, c\}$ is defined to mean

$$\{a, b, c\} = \omega = \frac{\sqrt{D} + b}{2c} = \frac{2a}{\sqrt{D} - b}$$

where

$$D = b^2 + 4ac,$$

and $a$, $b$, $c$ are ordinary integers. Thus $\omega$ is a definite root of the equation

$$c\omega^2 = a + b\omega\ ;$$

so that the author makes two alterations in the traditional notation of Gauss and Dirichlet. One of these, the substitution of $b$ for $2b$, needs no justification, and has, indeed, become almost inevitable; the reason for the change of sign in $c$ is less obvious, especially as on p. 390 the typical quadratic equation is written

$$A + B\Omega + C\Omega^2 = 0$$

with

$$B^2 - 4AC = D.$$

It is true that, in consequence of this additional modification, there is a trifling gain of typographical elegance; but this seems to be outweighed by other disadvantages.

This chapter concludes with the approximate calculation of roots by means of continued fractions, a brief discussion of rational roots, and a very meagre treatment of what is really a fundamental problem, namely the resolution of a polynomial with integral coefficients into its irreducible factors. It is true that a method is given which is theoretically sufficient, but this is quite useless in practice; while, on the other hand, the purely tentative method illustrated by an example can only be made conclusive by special artifices. Some account ought, we think, to have been given of Kronecker's algorithm (Crelle, vol. 92), which, although tedious, is really practicable, and has the advantage of giving a definite answer after a finite number of trials which may be estimated beforehand.

Chapter xii. contains the elementary theory of the roots of unity, primitive roots to a modulus, indices, quadratic residues, and the law of quadratic reciprocity: the proof of this last is the trigonometrical one of Eisenstein.

We now come to Book III., on "Algebraical Quantities," and here the essential and characteristic part of the work may be said to begin. The key-note is struck at the commencement of chapter xiii. by the definition of a numerical corpus (*Zahlkörper*). The notion of a corpus, which is of the most fundamental character, is due to Dedekind, and is as follows. Let us take a finite or infinite system of elements $\alpha$, $\beta$, $\gamma$, &c., concerning which nothing is assumed except that they can enter into rational combination according to the rules of ordinary algebra; then the totality of all rational functions of $\alpha$, $\beta$, $\gamma$, &c., except those which involve division by zero, constitutes a corpus, denoted by

$$\Omega(\alpha, \beta, \gamma \ldots).$$

The simplest corpus is that of all rational numbers. This is contained in every other corpus; for if $\omega$ be any element of the corpus, then by definition the corpus contains $\omega/\omega$, that is, unity; and from this all other rational numbers may be derived by rational operations only.

If the elements of a corpus are all numbers, it is called a numerical corpus; but the elements may be independent variables, or even variables subject to algebraical conditions.

If $\Omega(\alpha, \beta, \gamma \ldots)$ is any corpus, and $x$ any quantity not contained in it, the corpus $\Omega(x, \alpha, \beta, \gamma \ldots)$ is said to be derived from $\Omega(\alpha, \beta, \gamma \ldots)$ by the adjunction of $x$.

If $z$ is an undetermined variable, the polynomial

$$f(z) = a_0 z^m + a_1 z^{m-1} + \ldots + a_m$$

is said to be a function in $\Omega$ when all the coefficients $a_0$, $a_1, \ldots a_m$ belong to $\Omega$.

When $\Omega$ is given, we may, if we like, regard all the quantities belonging to it as rational: for this reason Kronecker calls a corpus a domain of rationality.

A function in $\Omega$ is reducible (in $\Omega$) if it can be resolved into the product of two functions in $\Omega$. A function which is irreducible in $\Omega$ may be reducible in a corpus derived from $\Omega$ by adjunction.

Let $f(z)$ be an irreducible function in $\Omega$, of the $n$th degree in $z$; then the equation

$$f(z) = 0$$

is assumed to have $n$ conjugate roots $z_1, z_2 \ldots z_n$. If $\Omega$ is a numerical corpus, the roots have actual numerical values; but this is really immaterial so far as the general algebraic theory of the corpus is concerned.

By the separate adjunction of the roots, we obtain the conjugate corpora $\Omega(z_1)$, $\Omega(z_2) \ldots \Omega(z_n)$, each of which is called an algebraical corpus of the $n$th degree. These conjugate corpora are not necessarily all different; they may, in fact, be all identical, and the corpus is then called a Galoisian or normal corpus. In this case all the roots $z_i$ are expressible as rational functions of any one of them, and this leads to the definition of a normal (or Galoisian) equation.

In the chapter we are now considering, the author proves the important theorem that the simultaneous adjunction of several algebraic quantities is equivalent to the adjunction of one only, provided that it be appropriately chosen; develops the distinction between primitive and imprimitive corpora; defines the Galoisian resolvent of an equation; and shows that the corpus $\Omega(z_1, z_2, \ldots z_n)$, derived from $n$ conjugate corpora, is normal. This last proposition is the master-key to the whole Galoisian theory. After this we have a discussion of the substitutions of a normal corpus; it is important to observe that their number is equal to its degree. Then comes a brief digression on the elements of the theory of permutation-groups; and this leads to the very important conclusion that the group of $\mu$ substitutions of a normal corpus which contains $n$ conjugate algebraical corpora is isomorphic with a certain group of permutations of $n$ things. This may be regarded as a group of permutations of the $n$ conjugate roots, and is then called the Galoisian group of the corresponding equation.

The problem of the algebraical solution of an equation ultimately depends upon the nature of its Galoisian group, or, which comes to the same thing, of its Galoisian resolvent. The degree of this resolvent is equal to $\mu$, the degree of the Galoisian group; if this is equal to $n\,!$ ($n$ being the degree of the proposed equation), the equation is said to have no *Affect.* So long as we confine ourselves to the corpus $\Omega(a_0, a_1, \ldots a_{n-1})$, the general equation

$$a_0 z^n + a_1 z^{n-1} + \ldots + a_{n-1} = 0$$

has no *Affect.* But, by the adjunction of an appropriate algebraical quantity, the Galoisian resolvent may become reducible, and then any one of its irreducible factors is a Galoisian resolvent in the new domain of rationality. The process may admit of repetition, and we may say that the problem ultimately consists in finding algebraical quantities of the simplest possible kind, so that by their successive adjunction we may obtain a series of resolvents of lower and lower degree. No such reduction can be effected by the adjunction of irrationalities which are not contained in the normal corpus $\Omega(z_1, z_2, \ldots z_n)$ which is derived from the original equation. After the proof of this very important result (p. 516), the chapter concludes with a further discussion of imprimitive groups, with special reference to the Galoisian resolvent.

Chapter xv., on cyclical equations, contains applications of the foregoing theory, and should be read concurrently with chapter xiv. by those to whom Galois's theory is new. First of all, the general cubic and biquadratic are solved by a direct application of group-theory; and we are then introduced to the theory of Abelian and cyclic equations. An Abelian equation is a normal equation whose Galoisian group is commutative. It is not necessarily irreducible; on the other hand, an irreducible equation with a commutative group is necessarily Abelian (p. 535).

A cyclical equation is one whose group consists of a single cyclical substitution and its powers; in other words, when its roots may be arranged in such an order that all cyclical functions of them are rational. The solution of any Abelian equation may always be reduced to that of a system of cyclic equations (§ 163). The chapter concludes with the solution of cyclic equations by means of Lagrange's resolvent.

Chapter xvi., on cyclotomy, gives the theory of Gauss's periods, of the auxiliary functions which Jacobi denotes by $\psi_\lambda(\alpha)$, of the solution of cyclotomic equations by their means, and of Gauss's sums. It concludes with applications to complex numbers of the forms $x + yi$, $x + y\rho$, $\rho$ being a cube root of unity. It may be observed that the properties of the numbers $\psi_\lambda(\alpha)$ are very fully treated in Jacobi's lectures on the theory of numbers. These are not included in his collected works; but MS. copies of Rosenhain's redaction of them may be picked up occasionally.

Chapter xvii. contains a series of remarkable propositions, which are proved with comparative ease by means of the foregoing theory. First of all it is shown that if P, the group of an equation, is reduced by the adjunction of the roots of an Abelian equation, P has a normal divisor Q, the index of which is a prime; and, conversely, if P has a normal divisor of this kind, P may be reduced in the manner stated. (By a normal divisor of P is meant a self-conjugate sub-group, or, as Klein calls it, an "ausgezeichnete Untergruppe.")

We then pass on to the theory of metacyclic equations. A metacyclic equation is defined as one whose complete solution may be made to depend on that of a series of cyclic equations. It is shown (p. 598) that the necessary and sufficient condition to be satisfied by a metacyclic equation is that there should exist a series of groups

$$\mathrm{P}, \mathrm{P}_1, \mathrm{P}_2, \ldots 1$$

(of which the first is the Galoisian group of the equation) such that each group is a normal divisor, with prime index, of the group immediately before it in the series.

It is subsequently proved (§ 180) that the group of a metacyclic equation of prime degree is linear; that is to say, if its roots are

$$x_1\, x_2 \ldots x_t \ldots x_p$$

the group consists of the $p(p-1)$ permutations of the suffixes defined by

$$t' \equiv at + b \pmod{p}$$

where $a$ may have any of the values $1, 2, \ldots (p-1)$, and $b$ any of the values $0, 1, 2 \ldots (p-1)$.

Conversely every irreducible equation of prime degree whose group is linear, is metacyclic (p. 615).

A function of the roots $x_t$ which is unaltered by the permutations of the linear group, and by these only, is called a metacyclic function. Every metacylic function is a root of a rational equation $\mathrm{F}(y) = 0$ of degree $(n-2)\,!$ and the function $y$ may be so chosen that this equation has no multiple roots. Assuming that $y$ has been so chosen, the necessary and sufficient condition that a given equation $f(x) = 0$ may be soluble by radicals is that the resolvent $\mathrm{F}(y) = 0$ has a rational root (p. 620). This very beautiful proposition is the complete answer to the question first definitely put by Abel, namely: What algebraical equations are soluble by radicals?

At the end of this chapter will be found a very interesting application to metacyclic quintics. Among other things it is shown that if $\lambda$ and $\mu$ are any two rational quantities, and

$$\alpha = \frac{5^5\mu^4\lambda}{(\lambda-1)^4(\lambda^2+6\lambda+25)}, \qquad \beta = \frac{5^5\mu^5\lambda}{(\lambda-1)^4(\lambda^2+6\lambda+25)}$$

the quintic

$$x^5 + \alpha x + \beta = 0$$

may be solved by radicals.

One problem still remains: the actual construction of all possible metacyclic equations. This is considered in chapter xviii., the last in vol. I. The complete solution was announced by Kronecker, in his usual oracular way, in the Berlin *Monatsberichte* for 1853 and 1856. Prof. Weber here supplies us with a demonstration, which reproduces in an improved and simplified form his own Marburg memoir of 1892. It is based upon the properties of Lagrange's resolvent; and although it is impossible to analyse it in detail, the final result may be stated.

"Every root $\xi$ of a metacyclic equation of prime degree $n$ may be expressed in the form

$$\xi = \mathrm{A} + \sum_{s=0}^{s=n-2} \mathrm{K}_s \tau_s^{r_{n-2}} \tau_{s+1}^{r_{n-3}} \ldots \tau_{s+n-2}^{r_0}$$

where A is a rational quantity, $\tau_s = \sqrt[n]{k_s}$, K a rational function of $k_s$ the form of which is the same for all values of $s$. The quantities $k_s$ are the roots, different from each other, and from zero, of a cyclical equation of degree $(n-1)$. The exponents $r_0, r_1, \ldots r_{n-2}$ are the least positive residues of $1, g, g^2, \ldots g^{n-2}$ to the modulus $n$; $g$ being a primitive root of $n$. The $n$ values of $\xi$, obtained from this formula by giving to the radicals $\tau_s$ their different values, are the roots of one and the same rational equation.

"Moreover every quantity $\xi$ of this form is the root of an equation of the $n$th degree rational in the corpus K, whose elements are the coefficients of the equation satisfied by $k_0, k_1 \ldots k_{n-2}$. This equation in $\xi$ is irreducible, except in the special case when one of its roots is rational." (With regard to the last clause, see the Corrigenda at the end of vol. II.) If the corpus K is real, the quantities $k$ are either all real or all imaginary (see p. 551); in the first case one root $\xi$ is real and the rest imaginary, in the second case all the roots $\xi$ are real.

The last five pages of this volume contain the explicit determination of all metacyclic quintic equations.

An account of vol. II., and a general review of the whole work, is reserved for another article.

G. B. M.

## OUR BOOK SHELF.

*Hours with Nature.* By Rambramha Sanyal, C.M.Z.S. Pp. 168. (Calcutta: Lahiri, 1896.)

AS a first attempt on the part of a native Indian naturalist to familiarise his countrymen with some of the leading facts in nature, and to cultivate in them the faculty of observation, this little volume is clearly entitled to a welcome at our hands. To criticise the English would obviously be unfair, seeing that the author is writing in a foreign tongue; and, indeed, if this be borne in mind, his work is worthy of all praise, so far as this point is concerned.

Babu Sanyal, as stated on the title-page, is Superintendent of the Calcutta Zoological Gardens, where he has many opportunities of observing the habits of animals but little known in Europe. One communication from his pen—on the habits of Bennett's Mungoose (*Cynogale bennetti*)—has already appeared in the London Zoological Society's *Proceedings;* and it is a matter for regret that other observations on the habits of animals under his charge have not found a place in this volume. As the book is essentially a medley, to give an account of its contents is somewhat difficult. It commences with the description of an excursion in Bengal, in the course of which we are introduced to the Gangetic dolphin, kingfisher, and various other birds, squirrels, and the famous Botanical Garden at Calcutta. Chapter ii. gives us a history of the former superintendents of that institution, mainly compiled from the official handbook. In the third chapter the author takes the misapplication of the term "mole" to the Indian musk-shrew as the text for a sermon on moles and shrews; while in the next section we pass to such a widely different subject as an aquarium and its denizens. Of the other chapters, we can only mention that the sixth describes the tour of a party of Bengalis round the Indian Museum, Calcutta, while the ninth relates to Indian snakes; perhaps the best in the book, the author emphasising the "rib-walking" character of these animals. If the book reaches a second edition, we would, however, advise him to study Mr. Boulenger's works, when he would probably amend his classification of reptiles, and point out some means of distinguishing between snakes and limbless lizards.

It may be hoped that this little work may succeed in its object of awakening a love of nature, and exciting observation among our Bengali fellow-subjects.

R. L.

*Elements of Astronomy.* By Sir Robert Ball, LL.D., F.R.S. New edition, thoroughly revised. Pp. 469. (London: Longmans, Green, and Co., 1896.)

THIS edition has been subjected by the author to a thorough revision, special attention having been paid to the last chapter, which deals exclusively with astronomical constants. These latter will be found most interesting and valuable to all classes of astronomical students, for besides being presented with the facts in each case, references as to the source of information are always added. Little need be said about the other chapters, as the revision seems to have been very thorough, although occasional omissions have been found. We may, however, mention that the illustration on page 44 seems to be rather out of date, and might have been changed for one more modern. In paragraphs relating to stellar classification, perhaps it would have been better to refer to a more recent classification, in which we have every reason to believe that stars do not simply decrease in temperature, but both increase and decrease. Putting aside these two minor details, the book will prove an excellent text-book for those wishing to acquire a knowledge of the more important problems relating to astronomy. The admirable index attached will be found most complete.

*Practical Work in Physics.* By W. G. Woollcombe, M.A., B.Sc. Part iii. Light and Sound. Pp. x + 94. (Oxford: Clarendon Press, 1896.)

THIS is the third part of a course of practical physics, the two previous ones dealing with heat and general physics. A fourth part, on electricity and magnetism, will complete the work. The optical experiments in the present volume illustrate photometry, reflection at plane and at spherical surfaces, refraction at plane surfaces, and through lenses. The experiments in sound demonstrate the laws of transverse vibrations of wires, velocity of sound through gases and through solids, and interference of sound-waves. Only inexpensive apparatus is needed in order to carry out the experiments described, and the instructions, both as to construction of apparatus and performance with it, are clear and practicable. Experiments in sound depend upon the physiological perception of tone, and some students are unable to accurately perform them. The number of students with no "ear for music" (musicians will, perhaps, pardon the designation of the twanging of a monochord as music) is, however, very small. Mr. Woollcombe says that only five per cent. of his students have so little musical sense that the experiments in sound he describes cannot be satisfactorily carried out by them; and his experience is about the same as that of most teachers and demonstrators of physics.

*Peasblossom. The Story of a Pet Plant.* By Caroline Pridham. Pp. 180. (London: John Heywood.)

TAKEN altogether, this book is very attractively written. The descriptions are couched in the simplest words, and no botanical terms are used without full explanation. Observation is the basis of the text, and the development of a plant is traced from germination upwards, all the parts and all the stages being considered. The book will interest young readers, and will encourage them to study the life-history of common plants; the knowledge they will thus gain from text and nature will be worth having.